Review of Dynamical Equations

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1 Equations of Motion

The equations of motion in a rotating Cartesian frame with z as vertical coordinate are:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{g} + \mathbf{F}_r \tag{1}$$

Or in component form:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_{rx} \tag{2}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_{ry} \tag{3}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz} \tag{4}$$

Here, $\mathbf{u} = (u, v, w)$ are the zonal, meridional, and vertical components of velocity; p is pressure; ρ is density; $f = 2\Omega\sin\varphi$ is the Coriolis parameter, where Ω is Earth's rotation rate, and φ is latitude; and $\mathbf{F}_r = (F_{rx}, F_{ry}, F_{rz})$ is the frictional force per unit mass. D/Dt is the material derivative, defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{5}$$

The continuity equation takes the form

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{6}$$

This set of equations for (u, v, w, ρ) is closed if we have a thermodynamic equation for p and closures for $\mathbf{F}_r$. For example, if we combine the ideal gas law and the first law of thermodynamics, we obtain the following two equations that close the set of equations:

$$\frac{D\theta}{Dt} = \frac{\theta}{T}\frac{\dot{Q}}{C_p}, \quad \rho = \frac{p}{RT}. \tag{7}$$

Here θ is the potential temperature, defined as

$$\theta = T\left(\frac{p_r}{p}\right)^{\kappa}, \tag{8}$$

where $\kappa = R/C_p$, with gas constant R and specific heat at constant pressure C_p; p_r is some reference pressure (usually the mean sea level pressure, or 1 bar = $10^5 Pa$ is used); $\dot{Q}$ is the heating rate per unit mass. In adiabatic motions, $\dot{Q} = 0$, thus θ is materially conserved.

Additionally, if we have a physical tracer ϕ, then its evolution is given by

$$\frac{D\phi}{Dt} = S_\phi \tag{9}$$

Here, S_ϕ is the source of ϕ. If ϕ is a conserved variable without external sources, then $S_\phi = 0$.

2   Boussinesq Approximation

In the Boussinesq approximation, we decompose the density and pressure as

$$\rho(x, y, z, t) = \rho_0 + \delta\rho(x, y, z, t) \tag{10}$$
$$p(x, y, z, t) = p_0(z) + \delta p(x, y, z, t) \tag{11}$$

and assume that $\delta\rho \ll \rho_0, \delta p \ll p_0$. Additionally, the reference values of ρ_0 and p_0 satisfy hydrostatic balance, such that

$$-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - g = 0. \tag{12}$$

The Boussinesq equations of motion are obtained by substituting the decomposition and neglecting any $\delta\rho$ and δp terms that appear with ρ_0 and p_0, leading to

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial\delta p}{\partial x} + fv + F_{rx}, \tag{13}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial\delta p}{\partial y} - fu + F_{ry}, \tag{14}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial\delta p}{\partial z} + b + F_{rz}. \tag{15}$$

The Boussinesq continuity equation is just

$$\nabla \cdot \mathbf{u} = 0. \tag{16}$$

The buoyancy b in the vertical momentum equation is defined as

$$b = -g\frac{\delta\rho}{\rho_0}. \tag{17}$$

Using the idealized gas law, we can write

$$p = \rho RT \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} = \frac{\delta p}{p_0} - \frac{\delta T}{T_0} \approx -\frac{\delta T}{T_0}, \tag{18}$$

$$\theta = T\left(\frac{p_r}{p}\right)^\kappa \quad \Rightarrow \quad \frac{\delta\theta}{\theta_0} = \frac{\delta T}{T_0} - \kappa\frac{\delta p}{p_0} \approx \frac{\delta T}{T_0}, \tag{19}$$

with the decompositions

$$T(x, y, z, t) = T_0(z) + \delta T(x, y, z, t), \tag{20}$$

$$\theta(x, y, z, t) = \theta_0(z) + \delta\theta(x, y, z, t). \tag{21}$$

Here we have assumed that $\delta p/p_0 \ll \delta T/T_0$. This means that density fluctuations are mainly due to the temperature fluctuations; contributions from pressure fluctuations are small. With this assumption, the buoyancy b takes the following form

$$b = -g\frac{\delta\rho}{\rho_0} = g\frac{\delta T}{T_0} = g\frac{\delta\theta}{\theta_0}. \tag{22}$$

3 Reynolds Averaging, and Reynolds Stress

With the Boussinesq approximation, if we further decompose the values of u, v, w into the (time or spatial) mean values $\overline{u}, \overline{v}, \overline{w}$ and the fluctuating values u', v', w', and using the fact that $\overline{(\cdot)'} = 0$, we can write prognostic equations for the mean motion. First we deduce the continuity equation

$$\nabla \cdot \mathbf{u} = \nabla \cdot \overline{\mathbf{u}} + \nabla \cdot \mathbf{u}' = 0. \tag{23}$$

By taking the average, we can find that both the mean and fluctuating components are zero, i.e.,

$$\nabla \cdot \overline{\mathbf{u}} = \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0, \tag{24}$$

$$\nabla \cdot \mathbf{u}' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \tag{25}$$

Then, for the equations of motion, let's start with the equation for the x component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \delta p}{\partial x} + fv + F_{rx}. \tag{26}$$

We may take the average directly and obtain

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = -\frac{1}{\rho_0}\frac{\partial \overline{\delta p}}{\partial x} + f\overline{v} + \overline{F}_{rx}. \tag{27}$$

Then, taking the mean of the fluctuating component of the continuity equation multiplied by u', we get

$$0 = \overline{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)u'} \tag{28}$$

$$= \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right) - \left(\overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}}\right) \tag{29}$$

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Adding this equation to the left hand side of equation (27), we get

$$\frac{\overline{Du}}{Dt} = \frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} \tag{30}$$

$$= -\frac{1}{\rho_0}\frac{\partial \overline{\delta p}}{\partial x} + f\overline{v} + \overline{F}_{rx} - \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right) \tag{31}$$

Similarly for the other component equations:

$$\frac{\overline{Dv}}{Dt} = \frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} \tag{32}$$

$$= -\frac{1}{\rho_0}\frac{\partial \overline{\delta p}}{\partial y} - f\overline{u} + \overline{F}_{ry} - \left(\frac{\partial \overline{v'u'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}\right) \tag{33}$$

$$\frac{\overline{Dw}}{Dt} = \frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} + \overline{w}\frac{\partial \overline{w}}{\partial z} \tag{34}$$

$$= -\frac{1}{\rho_0}\frac{\partial \overline{\delta p}}{\partial z} + \overline{b} + \overline{F}_{rz} - \left(\frac{\partial \overline{w'u'}}{\partial x} + \frac{\partial \overline{w'v'}}{\partial y} + \frac{\partial \overline{w'w'}}{\partial z}\right) \tag{35}$$

And for any tracer ϕ:

$$\frac{\overline{D\phi}}{Dt} = \frac{\partial \overline{\phi}}{\partial t} + \overline{u}\frac{\partial \overline{\phi}}{\partial x} + \overline{v}\frac{\partial \overline{\phi}}{\partial y} + \overline{w}\frac{\partial \overline{\phi}}{\partial z} = \overline{S}_\phi - \left(\frac{\partial \overline{u'\phi'}}{\partial x} + \frac{\partial \overline{v'\phi'}}{\partial y} + \frac{\partial \overline{w'\phi'}}{\partial z}\right). \tag{36}$$

The covariance terms on the right hand sides of the equations represent the average effects of turbulent fluxes on the mean momentum and tracer budgets. They are usually referred to as Reynolds stresses. It is clear from the equations that, to solve for the first-order moment ($\overline{\psi}$) equations, we need to know the second-order moments (covariance $\overline{\psi'\zeta'}$). Althought we can further deduce prognostic equations of second-order moments, it would then involve third-order terms that we need to parameterize. Eventually, we have to develop closure equations for the higher-order moments, for example, by eddy diffusion with parameterized eddy diffusivity.

4   Turbulent Kinetic Energy and its Prognostic Equation

One especially important combination of second moment terms is the mean turbulent kinetic energy $\overline{E}$, defined as

$$E = \frac{1}{2}(u'u' + v'v' + w'w'), \quad \overline{E} = \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}). \tag{37}$$

The prognostic equation for $\overline{E}$ is deduced from the prognostic equations of u', v', and w'. The deduction for the u-component is given below, and it is similar for the other components.

First, we subtract the mean equation (31) from the full equation (26) to get the prognostic equation for u':

$$\frac{\partial u'}{\partial t} + \left(\overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} + \overline{w}\frac{\partial u'}{\partial z}\right) + \left(u'\frac{\partial \overline{u}}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} + w'\frac{\partial \overline{u}}{\partial z}\right)$$

$$+ \left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}\right)$$

$$= -\frac{1}{\rho_0}\frac{\partial \delta p'}{\partial x} + fv' + \overline{F}'_{rx} + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right) \tag{38}$$

Multiplying this equation by u' and taking the average $\overline{(\cdot)}$, we get

$$\frac{1}{2}\frac{\partial \overline{u'^2}}{\partial t} + \left(\overline{u}\frac{\partial \frac{1}{2}\overline{u'^2}}{\partial x} + \overline{v}\frac{\partial \frac{1}{2}\overline{u'^2}}{\partial y} + \overline{w}\frac{\partial \frac{1}{2}\overline{u'^2}}{\partial z}\right) + \left(\overline{u'u'}\frac{\partial \overline{u}}{\partial x} + \overline{u'v'}\frac{\partial \overline{u}}{\partial y} + \overline{u'w'}\frac{\partial \overline{u}}{\partial z}\right)$$

$$+ \left(\overline{u'\frac{\partial \frac{1}{2}u'^2}{\partial x}} + \overline{v'\frac{\partial \frac{1}{2}u'^2}{\partial y}} + \overline{w'\frac{\partial \frac{1}{2}u'^2}{\partial z}}\right)$$

$$= -\frac{1}{\rho_0}\overline{u'\frac{\partial \delta p'}{\partial x}} + f\overline{u'v'} + \overline{u'F'_{rx}} \tag{39}$$

Note that the Reynold stresses in last term on the right hand side have dropped out. The equation can be rewritten in vector form as:

$$\frac{\overline{D}\frac{1}{2}\overline{u'^2}}{Dt} + \left(\overline{u'u'}\frac{\partial \overline{u}}{\partial x} + \overline{u'v'}\frac{\partial \overline{u}}{\partial y} + \overline{u'w'}\frac{\partial \overline{u}}{\partial z}\right) + \nabla \cdot \overline{(\mathbf{u}'u'^2/2)}$$

$$= -\frac{1}{\rho_0}\overline{u'\frac{\partial \delta p'}{\partial x}} + f\overline{u'v'} + \overline{u'F'_{rx}} \tag{40}$$

Similarly, the equations for $\frac{1}{2}\overline{v'^2}$ and $\frac{1}{2}\overline{w'^2}$ are as below:

$$\frac{1}{2}\frac{\overline{D}\overline{v'^2}}{Dt} + \left(\overline{v'u'}\frac{\partial \overline{v}}{\partial x} + \overline{v'v'}\frac{\partial \overline{v}}{\partial y} + \overline{v'w'}\frac{\partial \overline{v}}{\partial z}\right) + \nabla \cdot \frac{1}{2}\overline{(\mathbf{u}'(v'^2))}$$

$$= -\frac{1}{\rho_0}\overline{v'\frac{\partial \delta p'}{\partial y}} - f\overline{v'u'} + \overline{v'F'_{ry}} \tag{41}$$

$$\frac{\overline{D}\frac{1}{2}\overline{w'^2}}{Dt} + \left(\overline{w'u'}\frac{\partial \overline{w}}{\partial x} + \overline{w'v'}\frac{\partial \overline{w}}{\partial y} + \overline{w'w'}\frac{\partial \overline{w}}{\partial z}\right) + \nabla \cdot \overline{\left(\mathbf{u}'(\frac{1}{2}w'^2)\right)}$$

$$= -\frac{1}{\rho_0}\overline{w'\frac{\partial \delta p'}{\partial z}} + \overline{b'w'} + \overline{w'F'_{rz}} \tag{42}$$

Summing up the three component equations, we can get the TKE prognostic equation, which takes the following form:

$$\frac{\overline{DE}}{Dt} = S + B + T + \epsilon \tag{43}$$

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where S, B, T, ϵ are source terms representing shear production, buoyancy production, transport by turbulent motion, and viscous dissipation. They are defined as follows:

$$S = -\overline{u'\mathbf{u}'} \cdot \nabla \overline{u} - \overline{v'\mathbf{u}'} \cdot \nabla \overline{v} - \overline{w'\mathbf{u}'} \cdot \nabla \overline{w}, \tag{44}$$

$$B = \overline{w'b'} = g\frac{\overline{w'\theta'}}{\theta_0}, \tag{45}$$

$$T = -\frac{1}{\rho_0}\overline{u'\frac{\partial p'}{\partial x} + v'\frac{\partial p'}{\partial y} + w'\frac{\partial p'}{\partial z}} - \nabla \cdot \overline{\mathbf{u}'E}$$

$$= -\nabla \cdot \overline{\mathbf{u}'\left(\frac{p'}{\rho_0} + E\right)}, \tag{46}$$

$$\epsilon = \overline{u'F'_{rx}} + \overline{v'F'_{ry}} + \overline{w'F'_{rz}} = \overline{u'(\nu\nabla^2 u')} + \overline{v'(\nu\nabla^2 v')} + \overline{w'(\nu\nabla^2 w')}. \tag{47}$$

We have used $\delta p' = p', \delta\theta' = \theta'$, since the reference profiles $p_0(z), \theta_0(z)$ do not depend on time; ν is the kinematic viscosity coefficient. Shear production is positive when the turbulent momentum fluxes are directed down the gradient of mean momentum. Buoyancy production is positive when buoyant parcels move (and accelerate) upward or negatively buoyant parcels move (and accelerate) downward. The first term in the transport term T is related to the acceleration with the fluctuating pressure, i.e., pressure work that redistributes kinetic energy among different parts of the fluid; the second term in T is the flux of local TKE that also transports energy. Both terms are divergences of fluxes that would sum to zero by integration over a domain with vanishing fluxes at the boundaries (e.g., a vertical column with horizontal homogeneity). Viscous dissipation is the dissipation of smallest-scale TKE into viscous heat.

For a horizontally homogeneous boundary layer, we may neglect all horizontal transport terms, and we also neglect the gradient terms of $\overline{w}$ since $|\overline{w}| \ll |\overline{u}|, |\overline{v}|$, so that the S and T terms are approximated as

$$S \approx -\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z}, \tag{48}$$

$$T \approx -\frac{\partial}{\partial z}\left(\frac{\overline{w'p'}}{\rho_0} + \overline{w'E}\right). \tag{49}$$

The buoyancy production term B is the same as above. And if we further assume a constant viscosity ν and neglect the flux divergence term $\nabla \cdot (\overline{u'\nabla u'} + \overline{v'\nabla v'} + \overline{w'\nabla w'})$, then the dissipation term ϵ can be approximated as

$$\epsilon \approx -\nu[(\overline{\nabla u'})^2 + (\overline{\nabla v'})^2 + (\overline{\nabla w'})^2] \tag{50}$$

This form of ϵ implies that the viscous dissipation is always negative when shear (or velocity differences) occur in the flow, and that the smallest-scale eddies will be most efficiently damped by viscous dissipation, since smaller length scale with the same velocity implies larger shear.