

ON THE BOUSSINESQ APPROXIMATION FOR A COMPRESSIBLE FLUID

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ABSTRACT

The full, non-linear equations governing thermal convection in a compressible fluid have been re-examined in order to determine the conditions under which the Boussinesq approximation is applicable. These conditions are (a) the vertical dimension of the fluid is much less than any scale height, and (b) the motion-induced fluctuations in density and pressure do not exceed, in order of magnitude, the total static variations of these quantities. Under these conditions the equations are formally equivalent to those for an incompressible system when the temperature gradient is replaced by its excess over the adiabatic and C_p replaces C_v .

INTRODUCTION

In the study of problems of thermal convection it is a frequent practice to simplify the basic equations by introducing certain approximations which are attributed to Boussinesq (1903). The Boussinesq approximations can best be summarized by two statements: (1) The fluctuations in density which appear with the advent of motion result principally from thermal (as opposed to pressure) effects. (2) In the equations for the rate of change of momentum and mass, density variations may be neglected except when they are coupled to the gravitational acceleration in the buoyancy force.

For a liquid the approximations are quite natural ones to apply and Rayleigh (1916) used them in a formal analysis of Bènard cells. For the case of a compressible fluid Boussinesq attempted to justify the first assertion by noting that atmospheric pressure fluctuations are much too small to produce the observed density changes. But Jeffreys (1930) went further and suggested a formal justification for the applicability of the Boussinesq-Rayleigh equations. However, Jeffreys restricted his discussion to the study of infinitesimal, steady motions.

In the present note we re-examine the simplifying assumptions and restrictions which lead to the set of equations applicable to the study of convection in a narrow layer of compressible fluid. In order to make explicit order-of-magnitude comparisons, we have confined our analysis to the case where the equation of state may be adequately represented by the perfect gas law.

EQUATIONS

The equations of motion are

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - g \rho \mathbf{k} + \mu \nabla^2 \mathbf{v} + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{v}) \quad (1)$$

and

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \quad (2)$$

where \mathbf{k} is a unit vector in the vertical (z) direction. In obtaining equation (1) the second coefficient of viscosity has been related to μ by asserting that it is independent of the rate of compression. The coefficient μ is taken as constant.

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The equation for the rate of change of energy is

$$\rho C_v \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T + p \nabla \cdot \mathbf{v} = k \nabla^2 T + Q + \mu \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \frac{2}{3} \mu (\nabla \cdot \mathbf{v})^2, \quad (3)$$

where Q represents the radiation source and the coefficients, C_v and k , have been taken as constant.

The viscous dissipation terms contribute negligibly to the energy balance. This can be seen by noting that the convection of internal energy, $\rho \mathbf{v} \cdot \nabla e$, is large compared with $\mu \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v})$. Since the internal energy is of order v_k^2 , where v_k is the kinetic velocity and $\mu \sim \rho l |\mathbf{v}_k|$, where l is the mean free path, we must compare terms like $\rho (|\mathbf{v}| v_k^2 / L)$ and $\rho (l |\mathbf{v}_k| v^2 / L^2)$, where L is some macroscopic scale. Thus the ratio of the viscous dissipation term to the rate of change of internal energy is of the order of $|v| l / |v_k| L$, and we conclude that viscous dissipation will play no significant role in convection problems.

The equation of state for a homogeneous system (in the thermodynamic sense) is of the form

$$\rho = \rho(p, T) \quad (4)$$

THE BASIC APPROXIMATION

Let f represent any one of the state variables: density (ρ), temperature (T), or pressure (p). We shall express these in the form

$$f(x, y, z, t) = f_m + f_0(z) + f'(x, y, z, t), \quad (5)$$

where f_m is the (constant) space average of f ; f_0 is the variation in the absence of motion; and f' is the fluctuation resulting from motion. (For simplicity we have omitted cases where f_m and f_0 depend on t , which would occur if boundary conditions were time-dependent.) We may also introduce the scale heights,

$$D_f = \left| \frac{1}{f_m} \frac{df_0}{dz} \right|^{-1}. \quad (6)$$

The basic approximation which we shall apply is that the fluid be confined to a layer whose thickness, d , is much less than the smallest scale height, D_f . Thus if $D = (D_f)_{\min}$, we have

$$d \ll D \quad (7)$$

throughout the fluid.

In particular, condition (7) implies that $d/D \rho \ll 1$. On integrating this latter condition from the level of minimum to the level of maximum density within the layer, we conclude that

$$\frac{\Delta \rho_0}{\rho_m} \equiv \epsilon \ll 1, \quad (8)$$

where $\Delta \rho_0$ is the maximum variation of ρ_0 across the layer. Condition (8) is often the basic restriction imposed at the outset of convection studies.

The basic approximation, condition (7), is sufficient to produce the simplified Boussinesq-Rayleigh equations if only motions of infinitesimal amplitudes are considered. However, in non-linear investigations it is necessary to make the additional restriction that the motion-induced fluctuations do not exceed, in order of magnitude, the static variation. Thus we require that

$$\left| \frac{\rho'}{\rho_m} \right| \leq 0(\epsilon). \quad (9)$$

Condition (9) must be verified a posteriori from solutions of the problem. However, there seems to be no experimental evidence to indicate that ρ' ever exceeds $\Delta\rho_0$, so that, if the original restriction on $\Delta\rho_0$ is satisfied, one may generally proceed to apply the equations and expect to satisfy condition (9).

THERMODYNAMIC RELATIONS

Equation (4) may be expanded about ρ_m, p_m, T_m in a Taylor series, to yield

$$\rho = \rho_m \left[1 - \alpha_m (T - T_m) + K_m (p - p_m) + \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial^2 \rho}{\partial T^2} \right)_m \frac{(T - T_m)^2}{p} \right. \\ \left. + \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial^2 \rho}{\partial p \partial T} \right)_m (p - p_m) (T - T_m) + \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial^2 \rho}{\partial p^2} \right)_m (p - p_m)^2 + \dots \right], \quad (10)$$

where

$$\alpha_m = - \left(\frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_m, \quad K_m = \left(\frac{1}{\rho} \frac{\partial \rho}{\partial p} \right)_m. \quad (11)$$

If the fluid is an ideal gas, the series expansion becomes

$$\rho = \rho_m \left[1 - \frac{T - T_m}{T_m} + \frac{p - p_m}{p_m} + \left(\frac{T - T_m}{T_m} \right)^2 - \frac{1}{2} \left(\frac{p - p_m}{p_m} \right) \left(\frac{T - T_m}{T_m} \right) + \dots \right], \quad (12)$$

where, for brevity, we have omitted the (ordinarily) small corrections required by variation in mean molecular weight and by the effect of the radiation pressure.

From the restriction imposed in the preceding section, it follows that $[(T - T_m)/T_m]^2$ and $[(p - p_m)/p_m] [(T - T_m)/T_m]$ cannot be greater than $O(\epsilon^2)$. Hence, to order ϵ , we may write

$$\frac{\rho - \rho_m}{\rho_m} = - \frac{T - T_m}{T_m} + \frac{p - p_m}{p_m} \\ = - \alpha_m (T - T_m) + K_m (p - p_m), \quad (13)$$

with the further consequence that

$$\rho_0 = \rho_m (K_m p_0 - \alpha_m T_0) \quad (14)$$

and

$$\rho' = \rho_m (K_m p' - \alpha_m T'). \quad (15)$$

EQUATIONS OF MOTION

In the absence of motion, the vertical component of equation (1) becomes

$$\frac{\partial p_0}{\partial z} = - g \rho_m - g \rho_0. \quad (16)$$

If we introduce the hydrostatic relation into equation (1), we have

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p' - g \rho' \mathbf{k} + \rho \nu \nabla^2 \mathbf{v} + \frac{1}{3} \rho \nu \nabla (\nabla \cdot \mathbf{v}). \quad (17)$$

We may introduce equations (5) and (8) into the continuity equation to obtain

$$\nabla \cdot \mathbf{v} = - \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left(\epsilon \frac{\rho_0}{\Delta \rho_0} + \epsilon \frac{\rho'}{\Delta \rho_0} \right) + O(\epsilon^2). \quad (18)$$

Hence, to order ϵ , equations (17) and (18) may be written

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_m} \nabla p' - g\epsilon \frac{\rho'}{\Delta \rho_0} \mathbf{k} + \nu \nabla^2 \mathbf{v}, \quad (19)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (20)$$

In equation (19) we have retained the term $g\epsilon(\rho'/\Delta\rho_0)\mathbf{k}$ even though it contains ϵ as a factor. This procedure is clearly necessary if we are to study convection problems in the present approximation, and the following justification may be made.

The quantity $\partial\mathbf{v}/\partial t$ measures the characteristic acceleration of the fluid. Now the system is driven by fluctuations of the density field, and hence we must insist that the characteristic acceleration be of order $(g\epsilon\rho'/\Delta\rho_0)\mathbf{k}$. This in turn forces the conclusion that the acceleration of gravity is always much greater than the characteristic acceleration. We note that this is a statement concerning the relative size of the accelerations which occur in a convecting fluid and is not an assumption about the magnitude of g .

Equation (19) can be further simplified in the following way. The vertical component of equation (19) is

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w = -\frac{1}{\rho_m} \frac{\partial p'}{\partial z} - g\epsilon \frac{\rho'}{\Delta \rho_0} + \nu \nabla^2 w, \quad (21)$$

where w is the vertical velocity component. Using equation (15) we may write

$$\begin{aligned} -\frac{1}{\rho_m} \frac{\partial p'}{\partial z} - g\epsilon \frac{\rho'}{\Delta \rho_0} &= -\frac{1}{\rho_m} \frac{\partial p'}{\partial z} - g \frac{p'}{p_m} + g \frac{T'}{T_m} \\ &= -\frac{1}{\rho_m} \left(\frac{\partial p'}{\partial z} + \frac{1}{H} p' \right) + g \frac{T'}{T_m}, \end{aligned} \quad (22)$$

where

$$H = \frac{p_m}{g \rho_m}. \quad (23)$$

The quantity H is the thickness of a layer with uniform density and pressure varying from p_m at the bottom to zero at the top. From the definition of scale height and equation (16) it is readily seen that

$$H = D_p + 0(\epsilon); \quad (24)$$

and, since $\partial p'/\partial z \geq p'/d$, we conclude that p'/H is negligible compared to $\partial p'/\partial z$ in equation (23).

Therefore, equation (21) becomes

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w = -\frac{1}{\rho_m} \nabla p' + g \alpha T' \mathbf{k} + \nu \nabla^2 w. \quad (25)$$

The considerations leading to equation (25) have an interesting consequence. We have seen that the fluctuation buoyancy term is a necessary part of the convection problem. However, the contribution of p' to this term is of $0(d/H)$ as compared to the contribution of the temperature fluctuation. Therefore to $0(d/H)$ we may approximate equation (15) by

$$\frac{\rho'}{\rho_m} = -\frac{T'}{T_m} = -\alpha_m T', \quad (26)$$

which is equivalent to the first of the approximations mentioned in the introduction. It is of interest to note that equation (26) is a consequence of the dynamical argument.

We also note here that a similar argument does not apply to equation (14), since p_0 provides a major contribution to hydrostatic balance.

THE HEAT EQUATION

In the absence of motion, equation (3) becomes

$$k\nabla^2 T_0 + Q_0 = 0. \quad (27)$$

Hence the equation for the temperature fluctuation is

$$\rho_v C_v \left(\frac{\partial T'}{\partial t} + \mathbf{v} \cdot \nabla T \right) + p \nabla \cdot \mathbf{v} = k \nabla^2 T' + Q', \quad (28)$$

where

$$Q' = Q - Q_0. \quad (29)$$

The term $p \nabla \cdot \mathbf{v}$ is retained here because it is of the same order as the other terms in equation (28).

To verify this, we first note that p_0/p_m is of order d/H . Further, as mentioned earlier, we expect that $|p'/p_m| = O(\Delta p_0/p_m)$. Hence $p = p_m + O(d/H)$. Now, using equations (14), (18), and (26), we may write

$$p \nabla \cdot \mathbf{v} = p_m \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left(\frac{T_0 + T'}{T_m} - \frac{p_0}{p_m} \right), \quad (30)$$

where we have neglected terms of order d/H . But, as a consequence of equation (16),

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \frac{p_0}{p_m} = -\frac{w}{p_m} g \rho_m + O\left(\frac{d}{H}\right), \quad (31)$$

so that equation (30) becomes

$$p \nabla \cdot \mathbf{v} = \frac{p_m}{T_m} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (T_0 + T') + w g \rho_m. \quad (32)$$

Thus equation (28) is reduced to the form

$$\rho_m C_p \left(\frac{\partial T'}{\partial t} + \mathbf{v} \cdot \nabla T \right) + w g \rho_m = k \nabla^2 T' + Q', \quad (33)$$

where $C_p = C_v + p_m/\rho_m T_m \equiv C_v + R$. Equation (33) simplifies to the equation suggested by Jeffreys when non-linear terms and time derivatives are neglected. A more familiar form of the equation is

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T' + w \left(\frac{\partial T_0}{\partial z} + \frac{g}{C_p} \right) = K \nabla^2 T' + \frac{Q'}{\rho_m C_p}, \quad (34)$$

where $-g/C_p$ is the adiabatic gradient and

$$K = \frac{k}{\rho_m C_p}. \quad (35)$$

SUMMARY

It has been shown that the equations governing convection in a perfect gas are formally equivalent to those for an incompressible fluid if the static temperature gradient is replaced by its excess over the adiabatic, C_v is replaced by C_p , and the following approx-

imations are valid: (a) the vertical dimension of the fluid is much less than any scale height, and (b) the motion-induced fluctuations in density and pressure do not exceed, in order of magnitude, the total static variations of these quantities.

With these restrictions, the final system of equations is given by equations (20), (25), (26), and (34).

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