### Rayleigh Scattering Cross-Section

Electromagnetic waves drive oscillations of charges in matter, whose amplitude $x$ approximately satisfies the forced harmonic oscillator equation

$$\frac{d^2}{dt^2}x + \omega_0^2 x = \frac{F}{m},$$

where $\omega_0$ is the natural oscillation frequency, $F$ is the force acting on the charges, and $m$ is their mass. The force an electric field $E$ exerts on a charge $q_e$ is $F = q_e E$. For an incident electric field $E = \hat{E}\cos(\omega t)$ with frequency $\omega$, the solution for the charge oscillations is $x = \hat{x}\cos(\omega t)$, with amplitude

$$\hat{x} = \frac{q_e \hat{E}}{m(\omega_0^2 - \omega^2)}.$$

The oscillating charges in turn radiate electromagnetic waves. These are the scattered waves, and on average they carry away the power $P \propto q_e^2 \langle \ddot{x}^2 \rangle / (\epsilon_0 c^3)$, where $\langle \cdot \rangle$ indicates an average over a cycle, and the constant vacuum permittivity $\epsilon_0$ and speed of light $c$ ensure that the right-hand side has units of energy per unit time. So the scattered power depends on the squared acceleration $\ddot{x} = -\omega^2 \hat{x}\cos(\omega t)$ of the charges, or

$$P \propto \frac{q_e^2 \omega^4 \hat{x}^2}{\epsilon_0 c^3} = \frac{q_e^4 \hat{E}^2}{m^2 \epsilon_0 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}.$$

Because electromagnetic waves travel with the speed of light and have an energy density $\epsilon_0 \hat{E}^2$, the incident radiative energy flux is $\epsilon_0 c \hat{E}^2$. The ratio of the scattered power to the incident energy flux thus is

$$\sigma = \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}\sigma_T \quad \text{with} \quad \sigma_T \propto \left(\frac{q_e^2}{m \epsilon_0 c^2}\right)^2.$$

This scattering cross-section $\sigma$ expresses the scattered power in terms of the area of the incident beam that would have to be intercepted if all energy that impinges on that area were scattered. If $\omega \gg \omega_0$, the scattering cross-section $\sigma = \sigma_T$ is independent of frequency. This is the case of Thompson scattering on an unbound ($\omega_0 \to 0$) charge. If $\omega_0 \gg \omega$, the natural frequency is much higher than the frequency of the incident wave. In this case, the scattering cross section $\sigma = (\omega/\omega_0)^4 \sigma_T$ is a strong function of frequency $\omega$. This is the case of Rayleigh scattering on a bound charge. The Rayleigh scattering cross-section for macroscopic scatterers such as rain drops can also be expressed as $\sigma \propto \omega^4 r^6 \propto r^6/\lambda^4$ because both their total charge $q_e$ and mass $m$ scale with the volume $r^3$ of the scatterer.